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Constraint Factor of Notched Bars by d.c.-Board

NOBUO INOUE* AND HARRY C. STUTTS†
Purdue University, Lafayette, Ind.

A d.c.-board and its accompanying measuring instruments for solving statically determinate problems in plasticity are described. A radial flow problem, to which an exact solution is known, is used to test the theory basic to this approach, showing the validity of the theory. By applying the d.c.-board technique, the collapse load of notched bars in tension was estimated. The results are compared with theoretical results using truncated wedge stress field and with actual test data.

Nomenclature

a = width of notched section
b = half length of notch
d = width of bar
E = electric-field intensity
I = electric-current intensity
I_x, I_y = electric-current components across the bar thickness per unit normal length
k = yield stress in pure shear
L = load
R = resistance
V = voltage

Subscript

$*$ = a reference value

A D.C.-BOARD for plastic states of stress introduced in a previous paper[1] was set up at the Aerospace Sciences Laboratory of the School of Aeronautical and Engineering Sciences, Purdue University. It resembles a patch board used in conjunction with analog computers in that its sole purpose is that of a receptacle, holding variable potentiometers in a prescribed pattern. The board was set up in a rectangular pattern and fitted with jacks so the potentiometers could be set at a desired value and placed anywhere on the board. The potentiometers used fell into three categories: 0–15, 0–30, and 0–1000 kohm. The smaller potentiometers were used as the body of the geometric representation of the configuration being considered. As many as 180 of these potentiometers were in use at one time on one problem. The 0–30-kohm potentiometers were used for those points that required higher than 15-kohm resistance. These points were few and consisted mainly of the symmetry line where the normal resistance was doubled. The largest potentiometers were used to obtain satisfactory

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* Visiting Professor, School of Aeronautical and Engineering Sciences; on leave from Doshisha University, Kyoto, Japan. Member AIAA.

† Graduate Student, School of Aeronautical and Engineering Sciences; now with Analytic Services Inc., Bailey's Crossroads, Va.

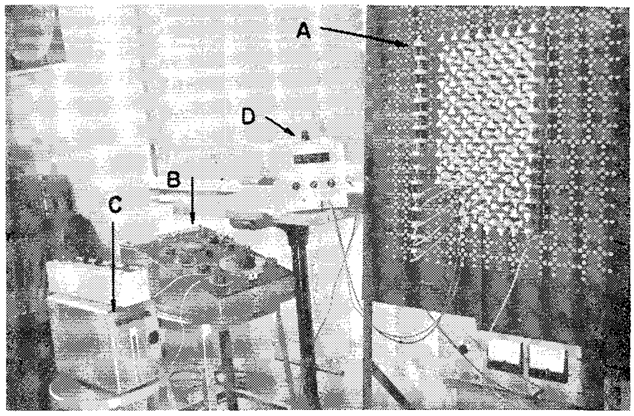

Fig. 1 Apparatus set up for a typical notched bar problem.

boundary conditions. Figure 1 shows the apparatus which is set up for a typical notched bar problem, and the following describes this apparatus. The potentiometers (A) were adjusted by means of a General Radio Company type 650-A impedance bridge (B) to which was connected an external Leeds and Northrup 2430-A galvanometer (C). This equipment gave a maximum error of 2% in setting the potentiometers. A Beckman model 4011 digital voltmeter (D) was used to measure the potential at different points on the d.c.-board. This instrument had an accuracy of 0.01%. The radial flow problem[2] in plasticity was used to test the validity of the d.c.-board techniques, since an exact solution is available for this problem. The maximum error encountered was 1.35%.

The notched bar problem was set up on d.c.-board. Because of symmetry, only one-fourth of the bar had to be placed on the d.c.-board, and the outside edge of the bar was connected to the positive side of 100 v while the centerline of the bar was grounded. Figure 1 shows a typical notched bar problem on the d.c.-board. The procedure for solving the problem was as follows. All potentiometers were initially set at 5 kohm except along the line of symmetry where the value was 10 kohm. The 1000-kohm potentiometers were adjusted, at near maximum value, so that a constant potential occurred across each of the 5-kohm potentiometers on the outside edge of the bar. This procedure resulted in a constant current I_* along the outside edge of the bar, which satisfied the boundary conditions. However, at the corner of the notch where two of the 5-kohm potentiometers join together at the boundary (since current is a vector), I_* had to be multiplied by $\frac{1}{2}^{1/2}$, and this new value of the current applied to each of these two potentiometers. Since the current was never measured directly but always was calculated from $I = E/R$, the voltage across these potentiometers was adjusted to be $\frac{1}{2}^{1/2}$ times the voltage across the rest of the outside potentiometers. Also, since the symmetry line had double resistance, the current across the outside potentiometer was equal to $I_*/2$. Once the boundary conditions had been set, neither the 1000-kohm potentiometers nor the potentiometers on the outside edge of the bar were ever adjusted again. The basic equations to the plane strain problem are given in Eq. (9) of the previous paper,[1] and it was found that by plotting the value of each potentiometer on Fig. 1 (field-intensity resistance relation) of the previous paper[1] a clearer picture resulted as to what adjustments needed to be made. R_* was fixed at 5 kohm, and I_* was a constant given by the boundary current. The resistance R was measured by the impedance bridge, thus leaving the current I as the only unknown. It must be remembered that current is a vector, and it has, therefore, both horizontal (I_x) and vertical (I_y) components, where $I = (I_x^2 + I_y^2)^{1/2}$. I_x is the current flowing through a potentiometer lying in the horizontal position, and I_y is the average current through the vertical potentiometers adjacent to it. A similar situation exists for a vertical potentiometer, where I_x is the average current in the horizontal potentiometers adjacent to it, and I_y is the current flowing through that particular vertical potentiometer. The current components in the vertical direction I_y are usually insignificant and can be neglected except near the corner of the notch. Since the main interest was in the doubled potentiometers across the centerline of the notch, more attention was paid to this area. The value of one potentiometer would be changed and it could be observed in the field-intensity resistance diagram, how this affected the other potentiometers. This amounted to a trial and error process. The problem was solved when all potentiometers fell exactly on the field-intensity resistance curve. This arrangement was never achieved, and the problem was assumed solved when the potentiometers were close to the curve. Any further adjustment of any potentiometer caused a divergence. This divergence was relative in that changing the value of a potentiometer caused some potentiometers to converge and others to diverge. Careful plotting was necessary to see how the overall system of potentiometers was acting and to find the values where no further change in any of the potentiometers was necessary. Through all adjustments the boundary conditions remained constant. Figure 2 shows a typical example of the values of the potentiometers obtained on the d.c.-board for the ratio of $2b/a = 0.2$.

The value of the load L acting on the bar that the resistances on the d.c.-board represented was calculated by

$$L = k\Sigma[\log(R/R_*) + 2]$$

along the symmetry line that was derived from Eq. (9) of the previous paper.[1] The constraint factors for notched bars of $a/d = \frac{5}{7}$ obtained from the d.c.-board are compared in Fig. 3 with theoretical results and measured values obtained by actual testing of a specimen. In the calculation of theoretical lower bounds for the constraint factor, the truncated wedge stress field introduced by Prager and Hodge[3] was used. The actual testing of the specimens was done on a 60,000-lb Tinius Olsen testing machine. Aluminum stock (Alcoa 6061-T6510) with a cross section of $\frac{3}{4} \times 2$ in. was used. The bars were loaded to 50,000 lb (slightly above the yield limit), before the notches were machined into them. After being notched the bars were again placed into the testing machine, and load vs strain curves were taken. The determination of $L/2ak$ was made in the following manner. L was taken as the collapse point for the notched bar on the load vs strain curve. It was determined by the method suggested by Watts and Ford.[4] The 50,000 lb that was initially placed on the unnotched bar was multiplied by the ratio of a/d and this value was used as $2ak$.

In concluding the present report, the authors would like to say that the d.c.-board method has an advantage over the lower bounds found by a truncated wedge stress field, and this is especially true in problems that have complex con-

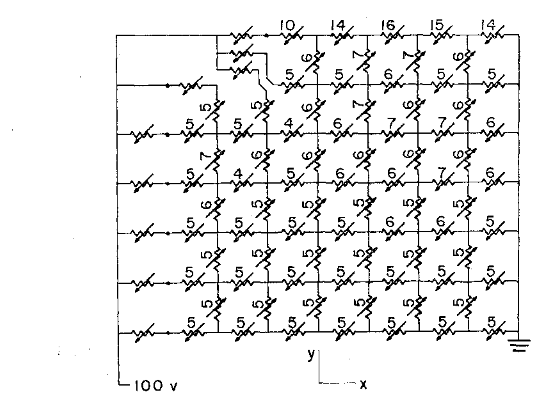

Fig. 2 Potentiometer values obtained by the d.c.-board theory in kilohms for $2b/a = 0.2$.

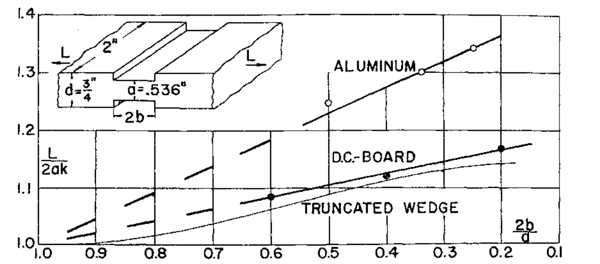

Fig. 3 Comparison of constraint factor of notched bars as obtained by d.c.-board (lower bound), truncated wedge stress field (lower bound), and actual testing.

figurations in which it is difficult to approximate the boundary between the elastic and inelastic regions.

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Thermal Choking of Partially Ionized Gases

K. T. Yen*
General Electric Company, Philadelphia, Pa.

THERMAL choking is a well-known phenomenon in classical aerodynamics.[1] More recently, Resler and Sears[2,3] investigated analytically channel flows with magnetogasdynamic effects and found increased possibilities of choking and smooth passage at the sonic speed. The purpose of this note is to show that for partially-ionized monatomic gases, a simple criterion such as "choking at Mach number equal to 1" cannot be obtained. Only equilibrium flows of monatomic gases with ionization fraction given by the Saha equation[4] will be considered.

The basic equations are

$$\frac{d}{dx}(nAu) = 0 \tag{1}$$

$$nu\frac{du}{dx} + R\frac{d}{dx}\{n(1+\alpha)T\} = 0 \tag{2}$$

$$nu^2\frac{du}{dx} + \frac{\gamma R}{\gamma - 1}nu\frac{d}{dx}\{(1+\alpha)T\} + nuRT_{\text{ion}}\frac{d\alpha}{dx} = q \tag{3}$$

$$\frac{\alpha}{1-\alpha^2}\frac{p}{p_0} = 3.16 \times 10^{-7}T^{2.5}\exp\left(-\frac{T_{\text{ion}}}{T}\right) \tag{4}$$

$$p = k(1+\alpha)nT \tag{5}$$

where n is the number density of the neutral monatomic gas, α is the number fraction of ionization, A is the area of the channel, q is the rate of heat energy density supplied to the gas, p_0 is the pressure of 1 atm, and T_{ion} is the ionization

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* Specialist, Aerodynamics Research, Space Science Laboratory, Missile and Space Division. Member AIAA.